Int. J. Solids Structures Vol. 20, No. 3, pp. 211-231, 1984
Printed in Great Britain.

0020-7683/84 $3.00 + .00
Pergamon Press Ltd.

OPTIMAL STRUCTURAL DESIGN UNDER MULTIPLE EIGENVALUE CONSTRAINTS

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(*Received 9 August 1982; in revised form 14 January 1983*)

Abstract—With increasing complexity, structures that are optimally designed subject to an eigenvalue constraint (buckling load, fundamental frequency, etc.) are likely to display multiple coincident eigenvalues. For example, optimal statically determinate and simply indeterminate columns buckle into a single mode, whereas fixed–fixed columns may exhibit a dual buckling mode. This phenomenon has been observed first by Olhoff and Rasmussen[1].

Even if the optimal design is in the interior of the admissible design space, multiple eigenvalue solutions are not stationary but singular. The nature of the singularity is the main topic of this investigation. Necessary and sufficient conditions for local and global optimality are explored and explicit optimality criteria are established for a double mode solution. In that case the space of all admissible design changes is split into a two-dimensional subspace in which the dual eigenvalues separate, and a complementary subspace in which they remain coincident. It is within this latter subspace that sequential approximations must take place.

The criteria developed here are applied first to a two-degree-of-freedom system. In addition, an exact analytical solution is established for the fixed–fixed column problem. The accuracy of the numerical solution of this problem in [1] had been challenged but is now confirmed.

The paper ends with a discussion of the ring buckling problem. The prismatic design is shown to be optimal without being stationary; however, unlike the cases discussed previously the optimal eigenvalue for the ring corresponds to a continuous spectrum of buckling modes.

1. INTRODUCTION

The issue of multiple eigenvalue constraints in connection with structural optimality has arisen only recently[1], yet it is rapidly developing into one of substantial concern. It is of major potential technical significance, as has been pointed out in [2] and as was further underlined by the number of formal presentations (e.g. [3–5]) and informal discussions at the NATO–NSF Advanced Study Institute on Optimization of Distributed Parameter Structures held in Iowa City in the spring of 1980. The issue came up again in general form at the 15*th Int. IUTAM Cong.* in Toronto in 1980[6], while related specific (and greatly simplified) problems have been discussed in [7] and elsewhere. In spite of these contributions a full understanding of the general nature of the effect of multiple eigenvalues in optimality theory, especially in relation to questions of necessity and sufficiency, appears to be lacking. The current study is therefore intended to help clarify some of the uncertainties that still remain.

Current concern with multiple eigenvalues is due to a discovery of Olhoff and Rasmussen, who considered the "best" design of a column under axial compression and with complete fixity at both ends. This problem had previously been considered by Tadjbaksh and Keller[8], who employed the calculus of variations and thereby obtained an "optimal" design which exhibits hinges at the quarter points and a symmetric buckling mode.

In their paper[1] Olhoff and Rasmussen noted that for a column so designed the actual buckling mode is antisymmetric and corresponds to an axial force which is much smaller than the one found in [8] (and is, in fact, much smaller than the smallest buckling load of a column of the same volume and constant cross-section). Moreover, they found that no other solution is satisfactory if it is based on the conventional calculus of variations approach as applied to an eigenvalue constraint. From this they drew the remarkably perceptive conclusion that the correct optimal solution is based on a double eigenvalue, and they in fact derived conditions of optimality and found a numerical solution to the problem. Note that the solution found by Olhoff and Rassmusen[1] exhibits no hinges.

The optimality conditions established in [1] were also derived through the use of the calculus of variations, but with the additional side condition of a dual eigenvalue. Questions regarding the validity of this derivation have been raised[9] on the basis that coincident multiple eigenvalues are not Fréchet-differentiable; moreover, since only first variations are involved in [1] the issue of the sufficiency of the results is still left open.

To place the matter in perspective we now restate and reformulate the single eigenvalue optimality problem in the Introduction (Section 1). An extension to multiple eigenvalues is then carried out in Section 2, and specific necessary and sufficient conditions for double eigenvalues, including a geometric interpretation, are derived in Section 3. Section 4 contains numerical and analytical solutions of some sample problems.

Let us start by considering the typical example of a conservative linear buckling problem, which is governed by the quadratic form

$$P_2(\mathbf{u}; H; \lambda) \equiv \int_\tau Q_2(\mathbf{u}; H)\,\mathrm{d}\mathbf{x} - \lambda \int_\tau W_2(\mathbf{u})\,\mathrm{d}\mathbf{x}, \tag{1}$$

in which Q_2 and W_2 are quadratic in the displacement vector function $\mathbf{u}(\mathbf{x})$, with $\mathbf{x} \in \tau$. The design is identified by $H(\mathbf{x})$, and the buckling load parameter by λ. The form Q_2 represents the strain energy density and is positive definite. If, as is assumed here, W_2 is also positive definite in $\mathbf{u}$, then all eigenvalues are non-negative, and each eigenmode $\mathbf{u}_i$ $(i = 1, 2, \ldots)$ is associated with an eigenvalue λ_i such that

$$P_{11}(\mathbf{u}_i, \mathbf{v}; H; \lambda_i) \equiv \int_\tau Q_{11}(\mathbf{u}_i, \mathbf{v}; H)\,\mathrm{d}\mathbf{x} - \lambda_i \int_\tau W_{11}(\mathbf{u}_i, \mathbf{v})\,\mathrm{d}\mathbf{x} = 0$$

$$i = 1, 2, \ldots \tag{2}$$

for all kinematically admissible displacement functions $\mathbf{v}$. We note, because of its paramount relevance to the topic of this paper, that although all modes $\mathbf{u}_i$ are distinct, the same may not be true of the associated eigenvalues λ_i.

Since P_{11}, Q_{11} and W_{11} are bilinear in $\mathbf{u}_i$ and $\mathbf{v}$, eqn (2) represents a linear eigenvalue problem. If all admissible function $\mathbf{u}(\mathbf{x})$ are normalized in the sense of

$$\int_\tau W_2(u)\,\mathrm{d}\mathbf{x} = 1, \tag{3}$$

then, for single eigenvalues λ_i, the corresponding eigenmodes $\mathbf{u}_i$ are uniquely determined (except for the sign). It may be noted that in certain types of eigenvalue problems, such as those dealing with vibrations, critical shaft speeds, etc. W_2 may also be a function of the design variable H. No essential difficulty is encountered in extending the results of this study to such problems.

The optimal design problem now consists in either finding a structure of minimum volume for given lowest eigenvalue constraint, or else to prescribe the volume and to design the structure so as to make the smallest eigenvalue as large as possible. The two problems can easily be shown to be equivalent. The second approach is adopted here, for convenience, although Olhoff and Taylor[10] have recently found it useful to employ the first approach in connection with optimal remodeling of structures under eigenvalue constraints.

If the modes $\mathbf{u}_i$ and associated eigenvalues λ_i are assumed to be Fréchet-differentiable[9] in the neighbourhood of a given design, with derivatives identified by superimposed dots (as in $\dot{\mathbf{u}}_i$), then differentiation of eqn (2) leads to

$$\int_\tau Q_{11}(\dot{\mathbf{u}}_i, \mathbf{v}; H)\,\mathrm{d}\mathbf{x} - \lambda_i \int_\tau W_{11}(\dot{\mathbf{u}}_i, \mathbf{v})\,\mathrm{d}\mathbf{x}$$

$$= -\int_\tau \frac{\partial Q_{11}}{\partial H}(\mathbf{u}_i, \mathbf{v}; H)\dot{H}\,\mathrm{d}\mathbf{x} + \dot{\lambda}_i \int_\tau W_{11}(\mathbf{u}_i, \mathbf{v})\,\mathrm{d}\mathbf{x} \quad \forall\, \mathbf{v}. \tag{4}$$

In view of eqn (2) the l.h.s. of eqn (4) represents a singular operator in $\dot{\mathbf{u}}_i$, and the inhomogeneous eqn (4) therefore has no solution unless its right side satisfies a certain orthogonality condition. The latter is obtained by subtracting eqn (4), with $\mathbf{v} = \mathbf{u}_i$, from eqn (2), in which we substitute $\dot{\mathbf{u}}_i$ for $\mathbf{v}$. With eqn (3) this leads to the familiar relationship

$$\dot{\lambda}_i = \int_\tau \omega_i \dot{H} \, \mathrm{d}\mathbf{x}, \tag{5a}$$

where

$$\omega_i \equiv \frac{\partial Q_2}{\partial H}(\mathbf{u}_i; H) \tag{5b}$$

has been shown previously[11] to represent the surface strain energy density in the critical "design fibers", i.e. in those fibers whose identity is affected by the choice of the value of the design variable H. For example, in a beam of depth H the critical design fibers are at the top and bottom of the beam.

Let the smallest eigenvalue be single, i.e. let

$$\lambda_1 < \lambda_2 \leqq \ldots; \tag{6}$$

then optimality requires, as a necessary condition, that

$$\dot{\lambda}_1 \leqq 0 \tag{7}$$

for all changes $\dot{H}$ which leave the volume V unchanged. Without significant loss of generality we may select H such that the condition of constant volume is represented by

$$\dot{V} \equiv \int_\tau \dot{H} \, \mathrm{d}\mathbf{x} = 0. \tag{8}$$

If $\dot{H}$ is unconstrained, or, equivalently, if the admissibility of any $\dot{H}(x)$ implies also the admissibility of $-\dot{H}(x)$, then the inequality in eqn (7) has to be ruled out since $\dot{\lambda}_1$ is linear in $\dot{H}$. In view of eqns (5), and by introducing, in the usual manner, a Lagrangian multiplier to account for the restriction of eqn (8) we arrive at

$$\omega_1 = k^2, \quad \mathbf{x} \in \tau \tag{9}†$$

as a necessary and sufficient condition for satisfying eqn (7) and hence as a necessary condition for local optimality.

The establishment of general global sufficiency conditions poses far greater problems, and no universal sufficiency conditions of technical usefulness appear to have been found thus far. However, a special case, but one of fairly broad technical importance, is listed in what follows.

Suppose we define a "stable" open domain $D_s[H_s(\mathbf{x}), \lambda_s]$ in the H, λ space in the sense that

$$P_2(\mathbf{U}; H_s; \lambda_s) \equiv \int_\tau Q_2(\mathbf{U}; H_s) \, \mathrm{d}\mathbf{x} - \lambda_s \int_\tau W_2(\mathbf{U}) \, \mathrm{d}\mathbf{x} > 0 \tag{10}$$

for all kinematically admissible functions $\mathbf{U}$. The domain D_s is bounded by the surface $S_1[H_1(\mathbf{x}; \lambda_1), \lambda_1]$, on which

$$\inf_{\mathbf{U}}{}' P_2(\mathbf{U}; H_1; \lambda_1) = P_2(\mathbf{u}_1; H_1; \lambda_1) = 0 \tag{11}$$

is satisfied. We note that eqn (11) leads to eqn (2) (with $i = 1$).

†Let the choice of the variable $H(\mathbf{x})$ in accordance with eqn (8) be such that the volume V increases with increasing value of H. For fixed mode $\mathbf{U}_i$ the strain energy density Q_2 must then increase, too, and ω_i is positive definite. This justifies the use of a positive Lagrangian multiplier.

Let us now consider a point $p_s(H_s, \lambda_s)$ in D_s, and an "unstable" point $p_u(H_u, \lambda_u)$ which lies "outside" of S_1. For p_u this implies that there exists at least one function $\mathbf{U}(\mathbf{x}) = \mathbf{U}_u(\mathbf{x})$, and a parameter λ_u, such that

$$\int_T Q_2(\mathbf{U}_u; H_u)\,d\mathbf{x} - \lambda_u \int_T W_2(\mathbf{U}_u)\,d\mathbf{x} < 0. \tag{12}$$

If we subtract eqn (12) from eqn (10) (with $\mathbf{U} = \mathbf{U}_u$), then

$$\int_T [Q_2(\mathbf{U}_u; H_s) - Q_2(\mathbf{U}_u; H_u)]\,d\mathbf{x} - (\lambda_s - \lambda_u) \int_T W_2(\mathbf{U}_u)\,d\mathbf{x} > 0. \tag{13}$$

As a special case let us now assume that Q_2 is concave in H, that is,

$$Q_2(\mathbf{U}; H) - Q_2(\mathbf{U}; H_0) \leqq \frac{\partial Q_2}{\partial H}{}_{(\mathbf{U};\, H_0)}(H - H_0) \tag{14}$$

for all $\mathbf{U}$, H and H_0. Many structural elements violate this inequality; however, the equality sign in eqn (14) is satisfied for the common case of a sandwich structure, which includes I-beams, sandwich plates and shells, etc. if the contribution of the web is ignored. If now eqn (14) (with $\mathbf{U} = \mathbf{U}_u$, $H = H_s$, $H_0 = H_u$) is substituted in eqn (13), then

$$\int_T \frac{\partial Q_2}{\partial H}{}_{(\mathbf{U}_u;\, H_u)}(H_s - H_u)\,d\mathbf{x} - (\lambda_s - \lambda_u) \int_T W_2(\mathbf{U}_u)\,d\mathbf{x} > 0. \tag{15}$$

Consider finally a point $p^*(H^*, \lambda^*)$ which represents an extension of the line $p_s - p_u$ beyond p_u, i.e.

$$\begin{aligned} H^*(\mathbf{x}) &= H_u(\mathbf{x}) + \alpha[H_u(\mathbf{x}) - H_s(\mathbf{x})] \\ \lambda^* &= \lambda_u + \alpha[\lambda_u - \lambda_s] \\ \alpha &> 0. \end{aligned} \tag{16}$$

Then

$$\begin{aligned} P_2(\mathbf{U}_u; H^*; \lambda^*) &\equiv \int_T Q_2(\mathbf{U}_u; H^*)\,d\mathbf{x} - \lambda^* \int_T W_2(\mathbf{U}_u)\,d\mathbf{x} \\ &\leqq \int_T Q_2(\mathbf{U}_u; H_u)\,d\mathbf{x} + \int_T \frac{\partial Q_2}{\partial H}{}_{(\mathbf{U}_u;\, H_u)}(H^* - H_u)\,d\mathbf{x} - \lambda^* \int_T W_2(\mathbf{U}_u)\,d\mathbf{x} \\ &= \left[\int_T Q_2(\mathbf{U}_u; H_u)\,d\mathbf{x} - \lambda_u \int_T W_2(\mathbf{U}_u)\,d\mathbf{x}\right] \\ &\quad - \alpha\left[\int_T \frac{\partial Q_2}{\partial H}{}_{(\mathbf{U}_u;\, H_u)}(H_s - H_u)\,d\mathbf{x} - (\lambda_s - \lambda_u) \int_T W_2(\mathbf{U}_u)\,d\mathbf{x}\right]. \end{aligned}$$

The inequality in this expression follows from inequality (14), and the equality is based on eqn (16). In the final expression the first bracket is postulated to be negative (see inequality (12)), and the second bracket is positive by inequality (15). With $\alpha > 0$ it follows that $P_2(\mathbf{U}_u; H^*; \lambda^*) < 0$, and p^* is therefore also unstable. In other words, a straight line from any point inside D_s to any point outside S_1, when extended beyond the latter, cannot enter D_s again. That is, D_s is *convex*, and the proof of the sufficiency of eqn (9) for global optimality is then obvious following standard reasoning.†

†It is of some interest to note that the domains, D_2, D_3, ... bounded by higher order surfaces S_2, S_3, ... (defining λ_2, λ_3, ...) are in general not convex because in that case eqn (10) applies only to restricted classes of functions $\mathbf{U}$. It can be shown, however, that D_2 is star-shaped with respect to all points in D_s.

If D_s is convex, then its intersection with the linear subspace

$$\int_T H(\mathbf{x})\,\mathrm{d}\mathbf{x} = V \tag{17}$$

(defining the volume constraint) is also convex. Because $V > 0$ the origin is not contained in that subspace. However, by selecting a reference design $H_0(\mathbf{x})$ (e.g. a prismatic design) and by introducing the translation

$$H(\mathbf{x}) = H_0(\mathbf{x}) + \bar{h}(\mathbf{x}), \quad \dot{H}(x) = \dot{\bar{h}}(\mathbf{x}), \tag{18}$$

where

$$\int_T H_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = V \tag{18a}$$

$$\int_T \bar{h}(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \tag{18b}$$

we establish the "homogeneous stable" domain $d_s(\bar{h}; \lambda_s)$ which is convex, which contains the origin, and whose bounding surface s_1 is concave toward the origin. It is noted that, for any arbitrary $h(\mathbf{x})$,

$$\bar{h}(\mathbf{x}) = h(\mathbf{x}) - h_{\text{average}} \tag{19}$$

satisfies eqn (18b).

The problem considered in [8] and again in [1] does not happen to have a concave strain energy density in the sense of (14), and the sufficiency condition therefore does not apply. Nevertheless this is not the reason why the results found in [8] are incorrect. In fact let us consider the much simpler, but equally demonstrative case of a fixed–fixed sandwich column of length l and of stiffness $EI = c^2(V/l)H(\xi)$, in which $c = \text{constant}$, $\xi = x/l$, and the design variable may be normalized in the sense of

$$\int_0^1 H\,\mathrm{d}\xi = 1.$$

With the axial force given by $N = \lambda c^2\ V/l^3$ the potential energy becomes

$$P_2(\mathrm{u}; \mathrm{H}; \lambda) = \frac{c^2 V}{2l^4}\left[\int_0^1 Hu''^2\,\mathrm{d}\xi - \lambda \int_0^1 u'^2\,\mathrm{d}\xi\right] \tag{20}$$

$$\left({}' \equiv \frac{\mathrm{d}}{\mathrm{d}\xi}\right)$$

which does satisfy eqn (14). The eigenvalue problem corresponding to eqn (2), with $i = 1$, is

$$(Hu_1'')'' + \lambda_1 u_1'' = 0, \quad \xi \in [0, 1] \tag{21}$$

$$u_1(0) = u_1'(0) = u_1(1) = u_1'(1) = 0,$$

and the optimality condition eqn (9) becomes

$$u_1''^2 = k^2, \quad \xi \in [0, 1] \tag{22}$$

in addition to the requirement[12] that u_1' be continuous at all hinge points ($H = 0$). The corresponding simply-supported column problem was solved by Prager and Taylor[13]. There the optimal design is symmetric, and the column buckles into a symmetric mode of constant curvature $u_1'' = k$, while the associated load parameter $\lambda_1 = 12$ represents an increase of slightly more than 20% over the corresponding value of $\lambda_{10} = \pi^2$ for the prismatic column of the same volume ($H_0 = 1$).

Similarly, the fixed–fixed column satisfying eqns (21) and (22) buckles into a symmetric mode, with $u_1'' = +k$ in the central half and $u_1'' = -k$ in the outer quarters, while the symmetric design $H_1(\xi)$ exhibits hinges at the quarter points. Once again, the buckling parameter $\lambda_1 = 48$ contrasts with $\lambda_{10} = 4\pi^2$ for the prismatic case $H_0(\xi) = 1$. In total analogy with the solution presented in [8], however, this is not an optimal solution, and the objections raised by Olhoff and Rasmussen in [1] apply with equal force to this case.†

This is easily visualized by considering the one-dimensional design subspace

$$H(\xi) = (1-\alpha)H_0(\xi) + \alpha H_1(\xi), \quad \alpha \in [0, 1],$$

in which the upper limit is imposed on α to avoid negative values of $H(\xi)$. The dependence of λ_1 on α is shown by the curve ABC in Fig. 1, with point A representing the prismatic column, and point C the "optimal" column designed according to eqns (21) and (22). Point C also corresponds to the solution presented in [8].

It turns out that, for increasing values of α, i.e. as increasing amounts of material are shifted from the region around the quarter points toward the center and the ends of the column, the resistance of the column against antisymmetric buckling is weakened. This is shown on the curve $A'BC'$, which represents the solution of eqn (21) for $i = 2$. The stable domain is represented by the area under ABC' and is in fact convex. The best design (within the subspace considered) is at point B and corresponds to a double eigenvalue. This phenomenon of a multiple eigenvalue solution is far from uncommon, and its likelihood increases with the complexity of the structure [2]. It represents an issue of considerable technical significance, and the establishment of the appropriate optimality conditions therefore occupies the remainder of the current investigation.

2. MULTIPLE EIGENVALUES

In what follows let us assume that the buckling condition eqn (2) is satisfied by the n-fold eigenvalue

$$\lambda_1 = \lambda_2 = \dots \lambda_n = \lambda \tag{23}$$

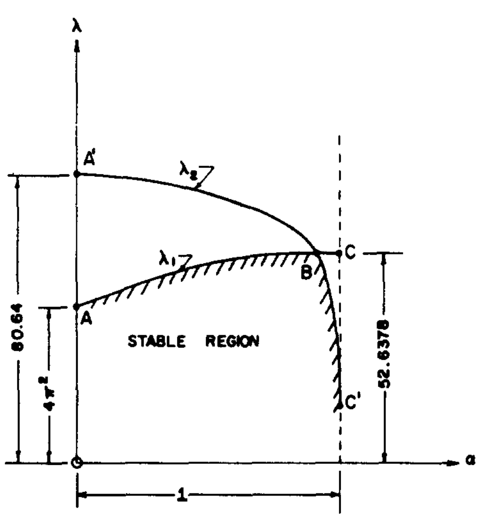

Fig. 1. Buckling load parameter λ vs design parameter α.

†These objections are mitigated, but not eliminated even if certain discontinuities in the slope are admitted [14].

in association with n linearly independent eigenfunctions $\mathbf{U}_i(x)$, $i = 1, 2, \ldots n$. Since any linear combination of these functions also satisfies eqn (2) it is possible, and turns out to be convenient, to orthonormalize these functions in the sense of

$$\int_\tau W_{11}(\mathbf{U}_i\mathbf{U}_j)\,\mathrm{d}\mathbf{x} = \delta_{ij}. \tag{24}$$

Note that eqn (24) do not determine the functions $\mathbf{U}_i$ uniquely. In particular, if $\mathbf{U}_i$ satisfies eqns (24), then any set of functions $\mathbf{u}_i$ defined by

$$\mathbf{u}_i(\mathbf{x}) = R_{ik}\mathbf{U}_k(\mathbf{x}) \qquad [R_{ik}]^{-1} = [R_{ki}] \tag{25}$$

also satisfies (24) if the rotation matrix is orthogonal as indicated. Moreover, the surface energy densities

$$\Omega_{ij} = \frac{\partial Q_2(\mathbf{U}_i\mathbf{U}_j)}{\partial H}, \quad \omega_{ij} = \frac{\partial Q_2(\mathbf{u}_i\mathbf{u}_j)}{\partial H}, \tag{26}$$

are readily seen to satisfy the transformation law

$$\omega_{ij} = R_{ik}R_{jl}\Omega_{kl}. \tag{27}$$

In deriving eqn (5), which identifies the "sensitivity" of the eigenvalue to design changes, no restriction was imposed on the possible multiplicity of the eigenvalues. Equation (5) therefore remains valid. In view of the multiplicity of the eigenfunctions $\mathbf{u}_i$ associated with the same eigenvalue, however, the question arises as to which function is to be used in computing $\dot{\lambda}_i$.

The answer to that question depends on $\dot{H}$, i.e. the "direction" of the design change. Consider, once again, eqn (2), with $i = I$ and $\mathbf{v} = \dot{\mathbf{u}}_J$, and subtract eqn (4), with $i = J$ and $\mathbf{v} = \mathbf{u}_I$. If I and J are two indices such that $\lambda_I = \lambda_J$ (eqn 23), and if we introduce, for assumed $\dot{H}$, the definitions (after returning to lower case subscripts)

$$\begin{aligned} \dot{\Lambda}_{ij}(\dot{H}) &= \int_\tau \Omega_{ij}\dot{H}\,\mathrm{d}\mathbf{x} \\ \dot{\lambda}_{ij}(\dot{H}) &= \int_\tau \omega_{ij}\dot{H}\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{28}$$

then

$$\dot{\lambda}_{ij} = R_{ik}R_{jl}\dot{\Lambda}_{kl} = \begin{cases} \dot{\lambda}_i & (i = j) \\ 0 & (i \neq j) \end{cases}. \tag{29}$$

In other words, eqn (5) is valid for the case of multiple eigenvalues provided the functions $\mathbf{u}_i$ have been so selected (or the given functions $\mathbf{U}_i$ have been so rotated) that for assumed $\dot{H}(x)$ the non-diagonal terms in $\dot{\lambda}_{ij}$ vanish.

It is important to note that whereas for given functions $\mathbf{U}_i$ the terms $\dot{\Lambda}_{ij}$ are linear in $\dot{H}$, the same is not true of the actual eigenvalue variations $\dot{\lambda}_i$ since the rotation matrices $[R_{ij}]$, in order to satisfy eqns (29), become themselves functions of $\dot{H}$. Nevertheless we record for future reference that

$$\dot{\lambda}_i(\alpha\dot{H}) = \alpha\dot{\lambda}_i(\dot{H}). \tag{30}$$

For optimality we now require, as a necessary condition, that

$$\inf_{i=1,2,\ldots n} \dot{\lambda}_i \leqq 0 \;\forall\, \dot{H} \in \text{eqn (8)}. \tag{31}$$

Unlike in eqn (7), however, the inequality sign has to be retained in eqn (31) since, with a replacement of $\dot{H}$ by $-\dot{H}$, the critical index i may shift.

The implementation of eqn (31) now proceeds in two stages. We first define a subspace $\dot{H}(\mathbf{x}) = \dot{h}_c(\mathbf{x})$ which satisfies eqn (8) and for which

$$\dot{\lambda}_1 = \dot{\lambda}_2 = \ldots \dot{\lambda}_n = \dot{\lambda}. \tag{32}$$

In view of eqn (29) it is clear that within this subspace the matrices $\dot{\Lambda}_{ij}$ must be proportional to the identity matrix, i.e.

$$\dot{\Lambda}_{ij}(\dot{h}_c) = \dot{\lambda}(\dot{h}_c)\delta_{ij}. \tag{33}$$

It is also noted that $\dot{\lambda}$ is linear in $\dot{h}_c$. If the latter is unconstrained, then, by the same argument as the one following eqn (7), the inequality in eqn (31) must now be ruled out. A necessary condition for optimality within the subspace $\dot{h}_c$ then is given by

$$\dot{\lambda}_i = 0, \quad i = 1, 2, \ldots n$$

or:

$$\dot{\Lambda}_{ij} = \int_\tau \Omega_{ij}\dot{h}_c \, d\mathbf{x} = 0, \quad i, j = 1, 2, \ldots n \tag{34}$$

$$\forall \dot{h}_c \in \int_\tau \dot{h}_c \, d\mathbf{x} = 0,$$

and, with the introduction of Lagrangian multipliers γ_{ij}, eqns (34) are satisfied provided

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}\Omega_{ij} = k^2, \quad \mathbf{x} \in \tau. \tag{35}$$

Equations (35) are necessary for local optimality and represent a generalization and extension of the conditions first introduced by Olhoff and Rasmussen[1]. A condition for global optimality is implied by convexity of D_s (see Section 1). Moreover, global optimality also imposes restrictions on the values of the Lagrangian multipliers γ_{ij}. These conditions have not been established thus far for the general case $n > 1$. In the following section, however, we derive specific restrictions for the case $n = 2$.

3. SPECIAL CASE—DOUBLE EIGENVALUE

For $n = 2$ the rotation matrix $[R_{ij}]$ reduces to

$$[R_{ij}] = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \tag{36}$$

and involves only the single rotation ϕ. Equations (29) simplify to

$$\tan 2\phi = \frac{2\dot{\Lambda}_{12}}{\dot{\Lambda}_{11} - \dot{\Lambda}_{22}}$$

$$\left.\begin{matrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{matrix}\right\} = \frac{\dot{\Lambda}_{11} + \dot{\Lambda}_{22}}{2} \pm \sqrt{\left(\frac{\dot{\Lambda}_{11} - \dot{\Lambda}_{22}}{2}\right)^2 + \dot{\Lambda}_{12}^2}. \tag{37}$$

If, analogously to the general case $n > 1$, the subspace $\dot{h}_c$ is defined by $\dot{\lambda}_1 = \dot{\lambda}_2$, then, by eqn (37), $\dot{h}_c$ satisfies

$$\dot{\Lambda}_{11} - \dot{\Lambda}_{22} = \int_\tau (\Omega_{11} - \Omega_{22})\dot{h}_c \, d\mathbf{x} = \int_\tau (\bar{\Omega}_{11} - \bar{\Omega}_{22})\dot{h}_c \, d\mathbf{x} = 0 \tag{38a}$$

$$\dot{\Lambda}_{12} = \int_\tau \Omega_{12}\dot{h}_c\,dx = \int_\tau \bar{\Omega}_{12}\dot{h}_c\,dx = 0 \tag{38b}$$

$$\int_\tau \dot{h}_c\,dx = 0 \tag{38c}$$

$$\dot{\lambda}_1 = \dot{\lambda}_2 = \int_\tau \bar{\Omega}_{11}\dot{h}_c\,dx = \int_\tau \bar{\Omega}_{22}\dot{h}_c\,dx \tag{38d}$$

where

$$\bar{\Omega}_{ij}(x) = \Omega_{ij}(x) - \frac{1}{\tau}\int_\tau \Omega_{ij}\,dx$$

$$= \Omega_{ij}(x) - (\Omega_{ij})_{\text{average}}. \tag{38e}$$

In eqns (38a), (b), and (d) Ω_{ij} and $\bar{\Omega}_{ij}$ may be used interchangeably in view of eqns (c) and (e).

Within the subspace $\dot{h}_c$, the optimality condition $\dot{\lambda}_1 = \dot{\lambda}_2 = 0$, subject to eqns (38), leads to the necessary condition

$$\gamma_{11}\Omega_{11} + 2\gamma_{12}\Omega_{12} + \gamma_{22}\Omega_{22} = k^2 \qquad x \in \tau \tag{39}$$

in conformity with the more general case covered in eqn (35). Equation (39) differs from the one presented in [1] through the presence of the term Ω_{12}. This issue is discussed later on in this section.

Since, by eqn (38a) and (38b), $\dot{h}_c$ is defined to be a subspace which is "orthogonal" to both $(\bar{\Omega}_{11} - \bar{\Omega}_{22})$ and $\bar{\Omega}_{12}$, the general expression for any permissible variation $\dot{H}(x)$ is given by

$$\dot{H}(x) = \alpha_1\dot{h}_1(x) + \alpha_2\dot{h}_2(x) + \dot{h}_c(x), \tag{40}$$

in which α_1 and α_2 are constants and in which

$$\dot{h}_1 = \frac{1}{2M_1}(\bar{\Omega}_{11} - \bar{\Omega}_{22})$$

$$M_1 = \frac{1}{2}\sqrt[+]{\left(\int_\tau (\bar{\Omega}_{11} - \bar{\Omega}_{22})^2\,dx\right)}$$

$$\dot{h}_2 = \frac{1}{M_2}\bar{\Omega}_{12} \tag{41}$$

$$M_2 = \sqrt[+]{\left(\int_\tau (\bar{\Omega}_{12}^2)\,dx\right)}$$

and in which further, by eqns (38),

$$\int_\tau \dot{h}_c\,dx = \int_\tau \dot{h}_c\dot{h}_1\,dx = \int_\tau \dot{h}_c\dot{h}_2\,dx = 0. \tag{42}$$

In other words, the general variation $\dot{H}$ is a linear combination of the two functions $\dot{h}_1$ and $\dot{h}_2$, each of unit norm, along which the eigenvalues separate, plus a complementary function $\dot{h}_c$ which is orthogonal to both $\dot{h}_1$ and $\dot{h}_2$ and along which the dual eigenvalue retains its duality.

It is convenient to select $U_1(x)$ and $U_2(x)$ in such a way that $\dot{h}_1$ and $\dot{h}_2$ are themselves mutually orthogonal in the sense that

$$\int_T \dot{h}_1\dot{h}_2\,dx = \int_T (\bar{\Omega}_{11} - \bar{\Omega}_{22})\bar{\Omega}_{12}\,dx = 0. \tag{43}$$

It is shown in the Appendix that this is always possible. If $\dot{H}$ as defined in eqns (40)–(42) is now substituted in eqn (37), then, in view of eqn (43),

$$\frac{\dot{\Lambda}_{11} - \dot{\Lambda}_{22}}{2} = \alpha_1 M_1 \qquad \dot{\Lambda}_{12} = \alpha_2 M_2.$$

$$\frac{\dot{\Lambda}_{11} + \dot{\Lambda}_{22}}{2} = \frac{\alpha_1}{4M_1}\int_T (\bar{\Omega}_{11} + \bar{\Omega}_{22})(\bar{\Omega}_{11} - \bar{\Omega}_{22})\,dx$$

$$+ \frac{\alpha_2}{M_2}\int_T \bar{\Omega}_{11}\bar{\Omega}_{12}\,dx + \int_T \bar{\Omega}_{11}\dot{h}_c\,dx,$$

and therefore

$$\left.\begin{matrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{matrix}\right\} = A_1\alpha_1 + A_2\alpha_2 \pm \surd(M_1^2\alpha_1^2 + M_2^2\alpha_2^2) + \int_T \bar{\Omega}_{11}\dot{h}_c\,dx, \tag{44}$$

where

$$A_1 = \frac{1}{4M_1}\int_T (\bar{\Omega}_{11} + \bar{\Omega}_{22})(\bar{\Omega}_{11} - \bar{\Omega}_{22})\,dx$$

$$A_2 = \frac{1}{M_2}\int_T \bar{\Omega}_{11}\bar{\Omega}_{12}\,dx = \frac{1}{M_2}\int_T \bar{\Omega}_{22}\bar{\Omega}_{12}\,dx$$

and in which M_1 and M_2 are defined in eqns (41).

Optimality requires that

$$\inf_{i=1,2} \dot{\lambda}_i \leqq 0 \quad \forall\, \alpha_1, \alpha_2, \dot{h}_c. \tag{45}$$

Equation (45) implies that $\dot{\lambda}_1$ and $\dot{\lambda}_2$ must not both be positive. Since, by eqn (30), the signs of both are reversed with the reversal of the sign of $\dot{H}$ it follows for the unconstrained case that a negative value for both $\dot{\lambda}_1$ and $\dot{\lambda}_2$ is also ruled out. Then eqn (45) is equivalent to [2]

$$\dot{\lambda}_1\dot{\lambda}_2 \leqq 0 \quad \forall\, \alpha_1, \alpha_2, \dot{h}_c. \tag{45a}$$

As pointed out before, within the subspace $\dot{H}(x) = \dot{h}_c(x)$ (i.e. for $\alpha_1 = \alpha_2 = 0$) eqn (45) or (45a) implies $\dot{\lambda}_1 = \dot{\lambda}_2 = 0$, or eqn (39), as a necessary condition of optimality. With the last term on the r.h.s. of eqn (44) thus eliminated the latter is reduced to

$$\left.\begin{matrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{matrix}\right| = A_1\alpha_1 + A_2\alpha_2 \pm \surd(M_1^2\alpha_1^2 + M_2^2\alpha_2^2). \tag{46}$$

The strict inequality in eqn (45) (or eqn 45a) is satisfied if

$$|A_1\alpha_1 + A_2\alpha_2| < \underset{+}{\surd}(M_1^2\alpha_1^2 + M_2^2\alpha_2^2) \quad \forall\, \alpha_1, \alpha_2$$

or, equivalently, if

$$M_1^2 - A_1^2 > 0$$
$$(M_1^2 - A_1^2)(M_2^2 - A_2^2) - A_1^2 A_2^2 > 0. \tag{47}$$

We note that the first inequality in (47) is already implied by the second.

A relationship between the constants A_i, M_i and the constants γ_{ij} appearing in eqn (39) is obtained by multiplying the latter by, respectively, $(\bar{\Omega}_{11} - \bar{\Omega}_{22})$ and $\bar{\Omega}_{12}$ and by integrating over the domain. With eqn (43) and some straightforward algebra this leads to

$$\frac{A_1}{M_1} = -\frac{\gamma_{11} - \gamma_{22}}{\gamma_{11} + \gamma_{22}}$$

$$\frac{A_2}{M_2} = -2\frac{\gamma_{12}}{\gamma_{11} + \gamma_{22}}$$

and the optimality condition eqn (47) reduces to

$$\gamma_{11}\gamma_{22} - \gamma_{12}^2 > 0. \tag{49}$$

Since the r.h.s. of eqn (39) has been postulated to be positive it will now be shown that γ_{11} and γ_{22} must also be positive. It is obvious from (49) that they cannot be of opposite sign; it therefore suffices to show they cannot both be negative.

In a previous footnote it is observed that ω (or Ω) is a positive definite quadratic form of its argument; hence

$$\Omega_{11} > 0; \quad \Omega_{22} > 0; \quad \Omega_{11}\Omega_{22} - \Omega_{12}^2 > 0.$$

These inequalities, in conjunction with (49), imply

$$\gamma_{12}\Omega_{12} \leq |\gamma_{12}|\,|\Omega_{12}| < \underset{+}{\sqrt{}}(\gamma_{11}\gamma_{22}\Omega_{11}\Omega_{22}).$$

Now let $\bar{\gamma}_{11} = -\gamma_{11} > 0$, $\bar{\gamma}_{22} = -\gamma_{22} > 0$; then the above inequality, when inserted into eqn (39), leads to the contradiction

$$-\bar{\gamma}_{11}\Omega_{11} + 2\sqrt{(\bar{\gamma}_{11}\Omega_{11}(\bar{\gamma}_{22}\Omega_{22})} - \bar{\gamma}_{22}\Omega_{22} \equiv -[\sqrt{(\bar{\gamma}_{11}\Omega_{11})} - \sqrt{(\bar{\gamma}_{22}\Omega_{22})}]^2 > k^2,$$

and

$$\gamma_{11} + \gamma_{22} > 0 \tag{50}$$

is therefore a necessary consequence.†

By setting

$$\frac{\gamma_{11}}{\gamma_{11} + \gamma_{22}} = 1 - \gamma \qquad \frac{\gamma_{12}}{\gamma_{11} + \gamma_{22}} = \beta \qquad \frac{k^2}{\gamma_{11} + \gamma_{22}} = \eta^2 \tag{51}$$

Equation (39) simplifies to

$$(1 - \gamma)\Omega_{11} + 2\beta\Omega_{12} + \gamma\Omega_{22} - \eta^2 = 0, \quad \mathbf{x} \in \tau \tag{52}$$

†Equations (49) and (50) suggest that the positive definiteness of $[\gamma_{ij}]$ may be required for optimality even for $n > 2$; our proof, however, is confined to $n = 2$.

or, equivalently, to

$$\left(1-\frac{A_1}{M_1}\right)\Omega_{11}-\frac{2A_2}{M_2}\Omega_{12}+\left(1+\frac{A_1}{M_1}\right)\Omega_{22}-2\eta^2=0, \tag{53}$$

and the optimality conditions eqns (49) and (50) take the simple form

$$\gamma(1-\gamma)-\beta^2>0 \tag{54}$$

which imply $0<\gamma<1$, $-\frac{1}{2}<\beta<\frac{1}{2}$.

It is noted that the strict inequality in eqn (54) assures local optimality in the $\dot{h}_1$, $\dot{h}_2$ subspace for which the two eigenvalues separate. A sufficiency condition for global optimality is obtained by adjoining the condition of convexity as discussed in Section 1 (see also [6]). It is also noted that the orthogonality condition eqn (43) was introduced only for computational convenience and is not necessary for the validity of either eqn (49) or (54), although, of course, it is essential if the computable constants A_i and M_i are to be used as well as for a geometric interpretation of these results (to be introduced at the end of this section).

In [1], as indicated before, the mixed term in eqn (52) is missing. Naturally, this may happen in specific cases, and can in any event be made to happen through the proper rotation of the modes $\mathbf{U}_i(x)$. In fact, with $\mathbf{U}^*=[R]\mathbf{U}$, the constants γ_{ij} in eqns (35) and (39) are transformed into the constants γ^*_{ij}, with their invariants preserved. Since γ_{ij} is symmetric it is of course always possible to find a rotation $[R]$ such that γ^*_{ij} becomes diagonal. In the special case of $n=2$ this happens, for example, when $H(\mathbf{x})$ and $\mathbf{U}^*_1(\mathbf{x})$ are symmetric and $\mathbf{U}^*_2(\mathbf{x})$ is antisymmetric.

In [1] these conditions are not fulfilled, nor do Olhoff and Rasmussen impose the condition of orthogonality, in the sense of eqn (24), on their buckling modes. In other words, the problem as posed in [1] differs from ours through the additional requirement $\gamma^*_{12}=0$ (in our notation), which is matched by a relaxation of the requirement of orthogonality. In what follows we show that their formulation is equivalent to ours in its full generality, and we derive sufficient conditions of optimality for their case.

If $\mathbf{U}^*_1$ and $\mathbf{U}^*_2$ are assumed to be functions of unit norm, as in [1], then they may in general be represented by

$$\begin{aligned} \mathbf{U}^*_1 &= \mathbf{U}_1\cos\phi^*_1+\mathbf{U}_2\sin\phi^*_1 \\ \mathbf{U}^*_2 &= -\mathbf{U}_1\sin\phi^*_2+\mathbf{U}_2\cos\phi^*_2. \end{aligned} \tag{55}$$

After inverting eqns (55) and substituting in eqn (39) we obtain the necessary condition

$$\gamma^*_{11}\Omega^*_{11}+2\gamma^*_{12}\Omega^*_{12}+\gamma^*_{22}\Omega^*_{22}=k^2, \quad x\in\tau \tag{56a}$$

where γ^*_{11} and γ^*_{22} are linear combinations of γ_{ij} and where

$$\begin{aligned} \gamma^*_{12}=\frac{1}{\cos(\phi^*_2-\phi^*_1)}\Big[&-\frac{\gamma_{11}+\gamma_{22}}{2}\sin(\phi^*_1-\phi^*_2)-\frac{\gamma_{11}-\gamma_{22}}{2}\sin(\phi^*_1+\phi^*_2) \\ &+\gamma_{12}\cos(\phi^*_1+\phi^*_2)\Big]. \end{aligned} \tag{56b}$$

The cross term γ^*_{12} in eqns (56) can be made to vanish through a proper choice of the rotation angles ϕ^*_i. With

$$\begin{aligned} \phi^*_1 &= \frac{1}{2}\tan^{-1}\frac{2\gamma_{12}}{\gamma_{11}-\gamma_{22}}+\psi+\bar{\psi} \\ \phi^*_2 &= \frac{1}{2}\tan^{-1}\frac{2\gamma_{12}}{\gamma_{11}-\gamma_{22}}+\psi-\bar{\psi} \end{aligned} \tag{57}$$

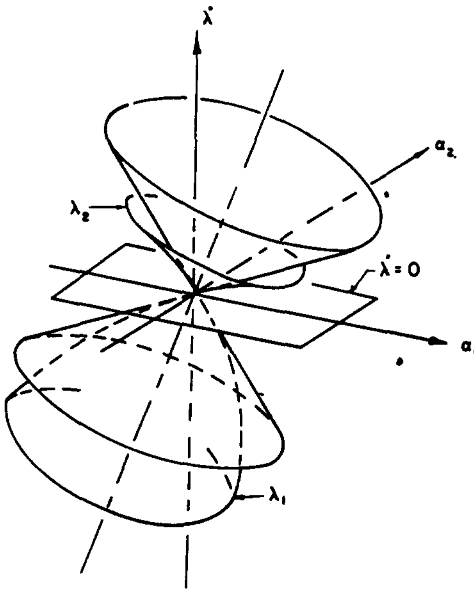

Fig. 2. λ-Surfaces and tangent cones near double eigenvalue singular point.

it follows that

$$\gamma^*_{12} = 0 \tag{58a}$$

when

$$\sin 2\bar{\psi} = \sqrt{\left(1 - 4\frac{\gamma_{11}\gamma_{22} - \gamma_{12}^2}{(\gamma_{11} + \gamma_{22})^2}\right)} \sin 2\psi \tag{58b}$$

Equations (58) define a class of admissible rotations ψ, $\bar{\psi}$. The invariants of $[\gamma_{ij}]$ transform into

$$\begin{aligned} \Delta_1 &= \gamma_{11} + \gamma_{22} = \gamma^*_{11} + \gamma^*_{22} + 2\gamma^*_{12} \sin 2\bar{\psi} \\ \Delta_2 &= \gamma_{11}\gamma_{22} - \gamma_{12}^2 = (\gamma^*_{11}\gamma^*_{22} - \gamma^{*2}_{12}) \cos^2 2\bar{\psi}, \end{aligned} \tag{59}$$

which must both be positive for optimality. If eqn (58) is satisfied, this reduces to

$$\begin{aligned} \Delta_1 &= \gamma^*_{11} + \gamma^*_{22} > 0 \\ \Delta_2 &= \gamma^*_{11}\gamma^*_{22} > 0, \end{aligned} \tag{60}$$

or $\gamma^*_{11} > 0$, $\gamma^*_{22} > 0$, as was already conjectured in [1].

In Ref. [2]* and elsewhere it has been assumed tacitly that the λ_1 and λ_2 surfaces intersect in a "curve". Equation (46) shows that this is not the case. In fact if λ_1 and λ_2 are plotted in a Cartesian subspace as functions of α_1 and α_2, with eqn (46) representing the tangent surface at the point being considered, then that surface turns out to be a double cone (see Fig. 2). The base of the cone can readily be shown to be an ellipse, and the optimality condition eqn (47) insures that the "horizontal" plane $\dot{\lambda} = 0$ does not intersect the cone except at the apex. If the λ_1-surface is convex it lies "inside" the lower cone, but the same restriction does not apply to the λ_2-surface (see footnote).

4. EXAMPLES

In our first example we re-examine the two-degree of freedom system (see Fig. 3) of Ref. [3]. The problem is to find the spring stiffnesses k_i which will make the fundamental frequency

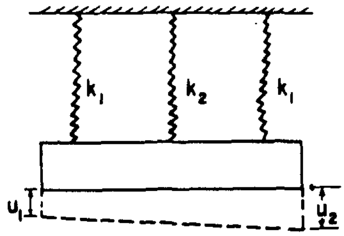

Fig. 3. Two degree of freedom system.

of vibration a minimum, subject to a cost constraint which is linearly related to the spring constants.

With u_1 and u_2 representing the displacements at the ends of the mass, and with $I = ml^2/12$, the system can readily be shown to be governed by

$$\begin{bmatrix} 4k_1 + k_2 & k_2 \\ k_2 & 4k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{61}$$

in which λ is proportional to the square of the frequency. The cost constraint and the physical constraints are, respectively,

$$\begin{aligned} J &= c_1 k_1 + c_2 k_2 \\ k_1 &\geqq 0, \quad k_2 \geqq 0, \end{aligned} \tag{62}$$

in which c_1 and c_2 are assumed to be positive. Equation (61) has the two solutions

$$\begin{aligned} \mathbf{u}^{(1)} &= \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \qquad \lambda_1 = 4k_1 \\ \mathbf{u}^{(2)} &= \frac{1}{\sqrt{6}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \lambda_2 = \frac{1}{3}(4k_1 + 2k_2), \end{aligned} \tag{63}$$

and the "energy density vectors" are given by

$$\{\omega_1\} = \begin{Bmatrix} 4 \\ 0 \end{Bmatrix}; \{\omega_{12}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}; \{\omega_{22}\} = \frac{1}{3} \begin{Bmatrix} 4 \\ 2 \end{Bmatrix}, \tag{64}$$

with

$$\{\omega_{ij}\} = \begin{Bmatrix} \dfrac{\partial Q_{11}(\mathbf{u}^{(i)}, \mathbf{u}^{(j)})}{\partial k_1} \\[2ex] \dfrac{\partial Q_{11}(\mathbf{u}^{(i)}, \mathbf{u}^{(j)})}{\partial k_2} \end{Bmatrix}.$$

Similarly to eqns (28) and (29) we have

$$\dot{\Lambda}_{11} = \dot{\lambda}_1 = \{\omega_{11}\}^T \begin{Bmatrix} \dot{k}_1 \\ \dot{k}_2 \end{Bmatrix} = 4\dot{k}_1 \tag{65a}$$

$$\dot{\Lambda}_{12} = 0 \tag{65b}$$

$$\dot{\Lambda}_{22} = \dot{\lambda}_2 = \{\omega_{22}\}^T \begin{Bmatrix} \dot{k}_1 \\ \dot{k}_2 \end{Bmatrix} = \frac{1}{3}(4\dot{k}_1 + 2\dot{k}_2) \tag{65c}$$

$$c_1\dot{k}_1 + c_2\dot{k}_2 = 0 \tag{65d}$$

in which the last equation follows from eqn (62).

Two cases may arise:

(1) $c_1 < 2c_2$. In this case the optimal solution is given by

$$k_2 = 0;\quad \dot{k}_2 > 0$$

$$k_1 = \frac{J}{c_1};\quad \dot{k}_1 = -\frac{c_2}{c_1}\dot{k}_2 < 0$$

$$\lambda_1 = \frac{4J}{3c_1};\quad \dot{\lambda}_1 = \left(-\frac{4}{3}\frac{c_2}{c_1} + \frac{2}{3}\right)\dot{k}_2 < 0$$

$$\lambda_2 = \frac{4J}{c_1} > \lambda_1.$$

The eigenvalues are distinct, and the optimality condition eqn (7) is satisfied.

(2) $c_1 \geq 2c_2$. In this case the solution given above is not optimal since $\dot{\lambda}_1 > 0$, nor is there any other optimal single eigenvalue solution. Optimality is reached with the double eigenvalue solution

$$k_1 = \frac{J}{c_1 + 4c_2}\qquad k_2 = \frac{4J}{c_1 + 4c_2}$$

$$\lambda_1 = \lambda_2 = \frac{4J}{c_1 + 4c_2}$$

$$\dot{\lambda}_1 = 4\dot{k}_1\qquad \dot{\lambda}_2 = \left(\frac{4}{3} - \frac{2}{3}\frac{c_1}{c_2}\right)\dot{k}_1$$

$$\dot{\lambda}_1\dot{\lambda}_2 = 4\left(\frac{4}{3} - \frac{2}{3}\frac{c_1}{c_2}\right)\dot{k}_1^2 \leqq 0$$

$$\inf_{i=1,2} \dot{\lambda}_i < 0\ \forall\ \dot{k}_1, \dot{k}_2 \in \text{eqn (65d)}$$

which satisfies eqns (45) and (45a).

The next example concerns the fixed–fixed column under axial force N, which was used by Olhoff and Rasmussen[1] to demonstrate the need for the introduction of multiple roots because the single-root solution found in Ref. [8] turned out to be incorrect. The solution found in [1] is numerical, and its accuracy has been challenged in [5], which is based on a different numerical scheme. In the current investigation we develop an analytical closed solution to this nonlinear problem.

The two coincident buckling modes u_i ($i = 1, 2$) are governed by

$$\begin{gathered}(EI\,u_{ixx})_{xx} + N\,u_{ixx} = 0 \qquad x \in [0, l]\\ u_i(0) = u_{ix}(0) = u_i(l) = u_{ix}(l) = 0.\end{gathered} \tag{66}$$

Let us assume, as in [1], [5] and [8], that, for given shape of the cross section, the moment of inertia I is proportional to the square of the cross section A. Let V represent the prescribed volume, and let us introduce the non-dimensional variables ξ and η and parameter λ through

$$\begin{gathered}I = c^2A^2 \qquad A(\xi) = \frac{V}{l}h(\xi) = \frac{V}{l}\sqrt{(\lambda)}\eta(\xi)\\ x = l\xi \qquad N = \lambda E c^2 V^2/l^4.\end{gathered} \tag{67}$$

Then eqn (66) reduces to

$$(\eta^2 u_i'')'' + u_i'' = 0, \quad \xi \in [0, 1] \qquad (i = 1, 2)$$
$$u_i(0) = u_i'(0) = u_i(1) = u_i'(1) = 0 \tag{68}$$
$$' \equiv \frac{d}{d\xi}.$$

In anticipation of the design variable $\eta(\xi)$ being symmetric with respect to the center of the column, and with the assumption that u_1 is symmetric and u_2 is antisymmetric, eqn (39) becomes

$$c_1{}^2\eta u_1''^2 + c_2{}^2\eta u_2''^2 = 1 \qquad \xi \in [0, 1] \tag{69}$$

in conformity with [1], while the volume constraint reduces to

$$\int_0^1 \eta \, d\xi = \frac{1}{\sqrt{\lambda}}. \tag{70}$$

Equation (69) is automatically satisfied by letting

$$u_1'' = \frac{\sin\theta}{c_1\sqrt{\eta}} \qquad u_2'' = \frac{\cos\theta}{c_2\sqrt{\eta}} \qquad \theta \equiv \theta(\xi), \tag{71}$$

which, when inserted into eqn (68), leads to

$$(\eta^{3/2}\sin\theta)'' + \eta^{-1/2}\sin\theta = 0 \qquad \xi \in [0, 1].$$
$$(\eta^{3/2}\cos\theta)'' + \eta^{-1/2}\cos\theta = 0. \tag{72}$$

By multiplying the first of these by $\cos\theta$ and the second by $\sin\theta$ and by taking the difference we obtain

$$3\eta^{1/2}\eta'\theta' + \eta^{3/2}\theta'' \equiv \eta^{-3/2}(\eta^3\theta')' = 0,$$

or, after integration,

$$\theta' = \frac{a}{\eta^3}, \tag{73}$$

in which "a" is a constant of integration. Similarly, by multiplying the first of eqn (72) by $\sin\theta$ and the second by $\cos\theta$, by adding the two, and by making use of eqn (73) we obtain

$$\eta'' + \frac{1}{2}\frac{\eta'^2}{\eta} - \frac{2}{3}\frac{a^2}{\eta^5} + \frac{2}{3\eta} = 0, \tag{74}$$

whose first integral is

$$\eta'^2 = \frac{4}{3}\frac{q(\eta)}{\eta^4} \tag{75a}$$

$$q(\eta) = -\eta^4 + \frac{3}{2}b\eta^3 - \frac{1}{3}a^2, \tag{75b}$$

in which "b" is another constant of integration. These constants are determined from the boundary conditions governing u_i and u_i' (eqns 68) after integrating the equations

$$u_1'' = -\frac{1}{c_1}(\eta^{3/2}\sin\theta)''$$

$$u_2'' = -\frac{1}{c_2}(\eta^{3/2}\cos\theta)'',$$

which follow from eqns (71) and (72). Eventually this leads to

$$u_1 = -\frac{1}{c_1}[\eta^{3/2}\sin\theta - \eta_0^{3/2}\sin\theta_0]$$

$$u_2 = -\frac{1}{c_2}[\eta^{3/2}\cos\theta + (2\xi - 1)\eta_0^{3/2}\cos\theta_0], \tag{76}$$

in which $\eta_0 \equiv \eta(0) = \eta(1)$ and $\theta_0 \equiv \theta(0)$. Note that the symmetry and anti-symmetry of u_1 and u_2, respectively, imply† $\theta(1) = 3\pi - \theta_0$.

The final form of eqns (75) and (73), respectively, is now

$$d\xi = \pm\frac{\sqrt{3}}{2}\frac{\eta^2\,d\eta}{\sqrt{(q(\eta))}} \tag{77}$$

$$d\theta = \pm\frac{\sqrt{3}}{2}\eta_0^3\sin 2\theta_0\frac{d\eta}{\eta\sqrt{(q(\eta))}} \tag{78}$$

where

$$q(\eta) = -\eta^4 + p\eta^3 - r \equiv (\eta_1 - \eta)(\eta - \eta_2)(\eta - \eta_3)(\eta - \eta_4)$$

$$p = \eta_0 + \frac{2}{3}\eta_0^3(1 + \cos 2\theta_0) \tag{79}$$

$$r = \frac{1}{3}\eta_0^6\sin^2 2\theta_0.$$

For $\theta_0 \neq (\pi/2)$ both p and r are positive, while the quadratic and linear terms in $q(\eta)$ are missing; this implies (see Fig. 4) that $\eta_1 > \eta_2 > 0$ and $\eta \in [\eta_2, \eta_1]$.

If ξ_2 is defined by $\eta(\xi_2) = \eta_2$ it follows from eqn (77) that

$$\xi_2 = \frac{\sqrt{3}}{2}\int_{\eta_2}^{\eta_0}\frac{\eta^2\,d\eta}{\sqrt{q}} \tag{80}$$

which can be expressed explicitly in terms of incomplete elliptic integrals of the first, second and third kind. Similarly, again from eqn (77)

$$\xi(\eta) = \xi_2 - \frac{\sqrt{3}}{2}\int_{\eta_2}^{\eta}\frac{\eta^2\,d\eta}{\sqrt{q}} \qquad \xi \in [0, \xi_2] \tag{81a}$$

$$\xi(\eta) = \xi_2 + \frac{\sqrt{3}}{2}\int_{\eta_2}^{\eta}\frac{\eta^2\,d\eta}{\sqrt{q}} \qquad \xi \in [\xi_2, 1/2]. \tag{81b}$$

†More generally, $\theta(1) = (2n+1)\pi - \theta_0$, $n = 0, 1, 2, \ldots$. The choice of $n = 1$ is one of convenience. It can readily be shown that the final result is independent of n.

To determine the two remaining constants η_0 and θ_0 we proceed as follows. We first note that $\xi(\eta_1) = (1/2)$; from eqn (81b), with the substitution of eqn (80), this means that

$$\frac{1}{2} = \frac{\sqrt{3}}{2}\left(\int_{\eta_2}^{\eta_0} + \int_{\eta_2}^{\eta_1}\right)\frac{\eta^2\,d\eta}{\sqrt{q}} \tag{82}$$

in which the notation is self-explanatory and in which the second integral is a complete elliptic integral. Moreover, since $\theta(0) = \theta_0$ and $\theta(1/2) = 3\pi/2$, integration of eqn (78) leads to

$$\frac{3\pi}{2} - \theta_0 = \frac{\sqrt{3}}{2}\eta_0{}^3 \sin 2\theta_0 \left(\int_{\eta_2}^{\eta_0} + \int_{\eta_2}^{\eta_1}\right)\frac{d\eta}{\eta\sqrt{q}} \tag{83}$$

which again involves elliptic integrals of the first, second, and third kind. Equations (82) and (83) are solved jointly† to determine η_0 and θ_0. Finally,‡ from the volume constraint eqn (70),

$$\frac{1}{\lambda} = 3\left\{\left(\int_{\eta_2}^{\eta_0} + \int_{\eta_2}^{\eta_1}\right)\frac{\eta^2\,d\eta}{\sqrt{q}}\right\}^2. \tag{84}$$

The results of these operations are given below:

$$\lambda = 52.3565$$

$$\theta_0 = 1.24984$$

$$\eta_0 = 0.18427 \qquad h_0 = 1.33334$$

$$\eta_1 = 0.18435 \qquad h_1 = 1.33392$$

$$\eta_2 = 0.03121 \qquad h_2 = 0.22583$$

$$\eta_{3,4} = -0.01523 \pm 0.02411i$$

and are incorporated in Fig. 4. Where applicable they are in close agreement with the numerical results obtained in [1]. The constants c_1 and c_2 can be computed on the basis of the normality conditions eqns (24).

It may be of some interest to note that the single mode solution found in [8] is contained in the general system presented here by setting $\theta_0 = \pi/2$. In that case $a = 0$ and $\theta' \equiv 0$, i.e. $\theta \equiv \pm\pi/2$ throughout (or $u_2'' = 0$). The quartic q reduces to $q = \eta^3\,(\eta_0 - \eta)$, with $\eta_1 = \eta_0$, $\eta_2 = \eta_3 = \eta_4 = 0$, while the integrals in eqns (80), (81) and (82), and (84) become elementary. Specifically, for $\xi \in [0, \xi_2 = 1/4]$ (and with the pattern repeated over the remainder of the column),

$$\xi = \frac{\sqrt{3}}{2}\int_{\eta}^{\eta_0}\sqrt{\left(\frac{\eta}{\eta_0 - \eta}\right)}\,d\eta = \frac{\sqrt{3}}{2}\eta_0\left[\frac{\pi}{2} - \sin^{-1}\sqrt{\left(\frac{\eta}{\eta_0}\right)} + \sqrt{\left(\frac{\eta}{\eta_0}\left(1 - \frac{\eta}{\eta_0}\right)\right)}\right] \tag{85}$$

and therefore, by setting $\eta = 0$ for $\xi = 1/4$, $\eta_0 = 1/\pi\sqrt{3}$. Equation (84) reduces to

$$\frac{1}{\lambda} = 12\left[\int_0^{\eta_0}\frac{\eta^{3/2}\,d\eta}{\sqrt{(\eta_0 - \eta)}}\right]^2,$$

†Although subroutines for all elliptic functions are available it was found more convenient to carry out the integrations in eqns (82) and (83) directly.

‡The separation of eqn (84) from eqns (82) and (83) is accomplished through the introduction of the design variable η in eqn (67). If, instead, h had been used, then η_0, θ_0, and λ would require the solution of three simultaneous equations.

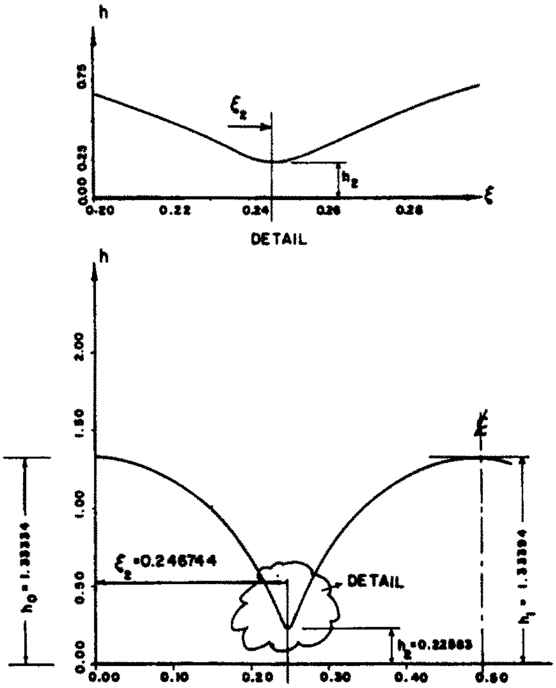

Fig. 4. Optimal design $h(\xi)$ for fixed–fixed column.

or

$$\lambda = \frac{16\pi^2}{3} \quad h_0 = \sqrt{(\lambda)}\, \eta_0 = \frac{4}{3}. \tag{86}$$

The difference between these values, which represent point C in Fig. 1, and the correct values corresponding to point B, is minuscule. A qualitative explanation for this phenomenon has been offered in [5] through the observation that shifting small amounts of material to the hinges ($\xi = 1/2$, $3/4$) greatly strengthens the column against buckling into the antisymmetric mode without materially reducing its resistance against buckling symmetrically. Figure 1 demonstrates this point graphically. Because of the discontinuity in the slope of the antisymmetric mode in the hinged column it is easy to show, from eqn (5), that the curve $A'BC'$ has a vertical tangent at point C'. Since the tangent to the curve ABC at point C is obviously horizontal, it follows that the correction introduced through the dual eigenvalue analysis is indeed small.†

The same process can be followed for the linear case in which $I = c^2A$, which applies to sandwich structures, idealized I-beams, etc. The procedure is similar, and with q reducing to a cubic polynominal the solution is again in terms of elliptic integrals. As pointed out previously, the single mode solution, similarly to the solution for the simply supported column [13], exhibits curvatures of constant magnitude and hinges at the quarter points and is associated with $\lambda = 48$ and $h_0 = 1.5$. For the double mode the calculations are not reproduced here because of their similarity to the case treated above. Moreover, with $\lambda = 47.956$ and $h_0 = 1.4988$ the solution is practically indistinguishable from the single mode solution.

5. ADDITIONAL AND CONCLUDING REMARKS

The purpose of the preceding sections has been to demonstrate, once again, the potential error in proceeding on the basis of the assumption of a single eigenvalue, and to investigate the nature of the singularity in the event of a multiple eigenvalue. Sufficiency conditions have been derived for the case of a double eigenvalue. In addition, previously established double eigenvalue solutions have been reinvestigated in the light of these singularity conditions, and an exact analytical solution has been obtained for the case of the optimal design of a fixed–fixed

†An "approximate" method introduced in [2] leads to results that are substantially in error. The method, which is based on replacing curves ABC and $A'BC'$ by second order parabolas, fails to take the vertical tangency at point C' into account.

column, for which numerical solutions (exhibiting slight discrepancies) had previously been found in [1] and [5].

Additional warning signals may sometimes be in order. Consider the case of a circular ring of radius R under constant external pressure, which is to be designed optimally against buckling. For the sake of simplicity, and without significantly abandoning generality, it is assumed that the cross section of the ring is of the sandwich type, with the moment of inertia I being proportional to the design variable h.

In nondimensional form the problem is governed by the quadratic form

$$P_2 = \frac{1}{2}\int_0^{2\pi} h(u''+u)^2\,d\phi - \frac{\lambda}{2}\int_0^{2\pi} (u'^2 - u^2)\,d\phi \tag{87}$$

in which u represents the radial displacement and a prime (as in u') designates differentiation with respect to the angle ϕ. For the prismatic design $h(\phi) = h_0 = 1$ the normalized buckling mode and lowest eigenvalue are given, respectively, by

$$u_{10} = \sqrt{\left(\frac{2}{3\pi}\right)}\cos 2(\phi - \phi_0) \qquad \lambda_{10} = 3 \tag{88}$$

where ϕ_0 is an arbitrary phase angle.

Assuming single mode optimality and setting $\phi_0 = 0$ we obtain, in line with eqn (9), the constant curvature solution [2]

$$\begin{aligned} u_1'' + u_1 &= \pm k, \quad \phi \in [0, 2\pi] \\ h(\phi) = 0 \quad &\text{for} \quad \phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \end{aligned} \tag{89}$$

which exhibits hinges at the four quarter points and is associated with $\lambda_1 = 3.653$. However, the "optimal" ring has now become a mechanism, which actually buckles under zero-pressure ($\lambda_2 = 0$). Has the value of the second eigenvalue again been reduced below that of the first eigenvalue, as was the case in the previous examples?

Clearly this has not happened here. Unlike the examples considered previously the present example displays a mechanism which "buckles" into a mode which is similar to the buckling mode of the prismatic ring, but rotated through 45°. In other words, the optimal design process, which was based (arbitrarily) on the assumption of $\phi_0 = 0$ has weakened the resistance of the ring against buckling into the same type of mode, but with a phase angle of $\phi_0 = \pi/4$.

Specifically, from eqns (5) and (88) (for arbitrary ϕ_0),

$$\dot{\lambda}_{10} = \frac{3}{2\pi}\int_0^{2\pi} \dot{h}\,[1 + \cos 4(\phi - \phi_0)]\,d\phi,$$

which, by invoking the constant volume condition eqn (8), reduces to

$$\dot{\lambda}_{10} = \frac{3}{2\pi}\int_0^{2\pi} \dot{h}\cos 4(\phi - \phi_0)\,d\phi. \tag{90}$$

Since ϕ_0 is arbitrary we may minimize (or maximize) $\dot{\lambda}_{10}$ by setting its derivative with respect to ϕ_0 equal to zero. This leads to

$$\begin{aligned} \genfrac{}{}{0pt}{}{\max}{\min}\,\dot{\lambda}_{10} &= \pm\frac{3}{2\pi}\sqrt{(S^2 + C^2)} \qquad \tan 4\phi_0 = \frac{S}{C} \\ S &\equiv \int_0^{2\pi} \dot{h}\sin 4\phi\,d\phi \qquad C = \int_0^{2\pi} \dot{h}\cos\phi\,d\phi, \end{aligned} \tag{91}$$

or, as a consequence,

$$\inf_{\phi_0} \dot{\lambda}_{10} \leqq 0 \quad \forall \dot{h}(\phi). \tag{92}$$

In other words, no modification of the prismatic design can lead to an increase of the smallest eigenvalue. Since for the sandwich cross section Q_2 is concave (see the equality in eqn 14) it follows, perhaps not unexpectedly, that the prismatic design of the ring is optimal (without being stationary). It may be conjectured that the same is true for other types of cross sections, but proof appears to be lacking.

Acknowledgements—This study has been supported by the National Science Foundation through Grant No. CEE-8110784 to the University of Illinois at Chicago (UIC), and by the Computing Center at UIC. The author also gratefully acknowledges the help he has received from Nihal Somaratna in carrying out the computations.

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APPENDIX

In the Appendix we show that it is always possible to select the functions $U_1(\mathbf{x})$, $U_2(\mathbf{x})$ in such a way that $\dot{h}_1(\mathbf{x})$ and $\dot{h}_2(\mathbf{x})$ are mutually orthogonal in the sense of eqn (43). In fact let U_1^* and U_2^* be two functions satisfying the orthonormality condition eqn (24). Then, as pointed out previously, the transformation

$$\begin{aligned} U_1 &= U_1^* \cos\theta + U_2^* \sin\theta \\ U_2 &= -U_1^* \sin\theta + U_2^* \cos\theta \end{aligned} \tag{A1}$$

leads to two functions which also satisfy eqn (24) for any value of θ. Moreover,

$$\begin{aligned} \bar{\Omega}_{11} - \bar{\Omega}_{22} &= (\bar{\Omega}^*_{11} - \bar{\Omega}^*_{22}) \cos 2\theta + 2\bar{\Omega}^*_{12} \sin 2\theta \\ 2\bar{\Omega}_{12} &= -(\bar{\Omega}^*_{11} - \bar{\Omega}^*_{22}) \sin 2\theta + 2\bar{\Omega}^*_{12} \cos 2\theta \end{aligned} \tag{A2}$$

in which $\Omega^*_{ij} = \dfrac{\partial}{\partial h} Q_{11}(U_i^* U_j^* h)$.

Equation (43) is satisfied, i.e.

$$\int_\tau (\bar{\Omega}_{11} - \bar{\Omega}_{22})\bar{\Omega}_{12}\, \mathrm{d}x = \int_\tau \dot{h}_1 \dot{h}_2\, \mathrm{d}x = 0, \tag{A3}$$

provided

$$\tan 4\theta = 4 \frac{\int_\tau (\bar{\Omega}^*_{11} - \bar{\Omega}^*_{22})\bar{\Omega}^*_{12}\, \mathrm{d}x}{\int_\tau [(\bar{\Omega}^*_{11} - \bar{\Omega}^*_{22})^2 - 4\bar{\Omega}^{*2}_{12}]\mathrm{d}x}. \tag{A4}$$

Note that θ is indefinite if both numerator and denominator vanish. Otherwise eqn (A4) has four solutions differing by multiples of $\pi/4$, with a rotation through $\pi/4$ representing an interchange between $\bar{\Omega}_{11} - \bar{\Omega}_{22}$ and $2\bar{\Omega}_{12}$, while $\bar{\Omega}_{11} + \bar{\Omega}_{22}$ remains invariant.